

Superfluid Phase Transition with Activated Velocity Fluctuations: Renormalization Group Approach

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(Dated: December 18, 2015)

A quantum field model that incorporates Bose-condensed systems near their phase transition into a superfluid phase and velocity fluctuations is proposed. The stochastic Navier-Stokes equation is used for a generation of the velocity fluctuations. As such this model generalizes model F of critical dynamics. The field-theoretic action is derived using the Martin-Siggia-Rose formalism and path integral approach. The regime of equilibrium fluctuations is analyzed within perturbative renormalization group method. The double (ϵ, δ) -expansion scheme is employed, where ϵ is a deviation from space dimension 4 and δ describes scaling of velocity fluctuations. The renormalization procedure is performed to the leading order. The main corollary gained from the analysis of the thermal equilibrium regime suggests that one-loop calculations of the presented models are not sufficient to make a definite conclusion about the stability of fixed points. We also show that critical exponents are drastically changed as a result of the turbulent background and critical fluctuations are in fact destroyed by the developed turbulence fluctuations. The scaling exponent of effective viscosity is calculated and agrees with expected value $4/3$.

I. INTRODUCTION

Non-equilibrium physics [1, 2] constitutes an interesting research topic to which a lot of effort has been devoted in last decades. In general such problems are difficult to solve exactly. However, a great simplification is possible near continuous phase transitions where new symmetry related to scale invariance appears. An immediate hallmark of it is divergence of the correlation length, which results into an importance of fluctuations on all length scales. The system then effectively forgets about microscopic details and can be described by a few coarse grained quantities.

The liquid-vapour critical point, λ -transition in superfluid helium ^4He and transition between ferro- and paramagnetic phase near Curie temperature in ferromagnetic materials belong to famous examples of continuous phase transitions. From an experimental point of view [3] a special role is devoted to the phase transition of ^4He where one can approach criticality closer than in any other system. However, special techniques have to be applied in order to overcome slow dynamic transitions. Interests in the theoretical investigation of Bose-condensates in the superfluid state retrieve attention after recent experimental achievements in condensation of diluted inert gases [4]. Superfluidity appears at the phase transition lambda point, where the viscosity of the fluid vanishes [5, 6]. However, the critical dimension of the viscosity coefficient, i.e. the law defining its behavior in the limit $\nu \rightarrow 0$, is not determined yet. This drawback is explained by the fact that traditionally the critical dynamics in the vicinity of the lambda point is described by model E or F in the standard terminology [7]. In the framework of a

traditional construction of aforementioned models the velocity, whose dynamics is described by the Navier-Stokes equation, is infrared (IR) irrelevant. As a result, the viscosity drops out and clearly cannot be analyzed.

From classical theory of fluids [8, 9] it is well-known that vanishing viscosity leads to the phenomenon of turbulence. A genuine property of turbulent flows with continuous phase transition is scale invariance. In an inertial interval power laws are generically observed and independent of viscosity (second Kolmogorov hypothesis). These findings are corroborated in the celebrated Kolmogorov works [10, 11].

A phenomenon of turbulence has been discovered and lately analyzed also in other than its original context, among others in high energy physics [12–19], inflation cosmology [20–22], ultracold gas [23] or quantum turbulence (QT) [24–28]. Due to a progress in experimental methods the latter was studied also directly [29–32]. For the turbulence in superfluids the quantum effects are of utmost importance. From a theoretical point of view the zero temperature (ground state) of bosonic superfluid is well described by the non-linear Schrödinger equation, also none as Gross-Pitaevskii equation [33, 34]. Different theoretical techniques [35] can be employed in order to study QT. Various aspects in connection with turbulence were recently analyzed [26, 36–38].

A common feature of all these studies is a concentration on the quantum state of the superfluid state and thus operate well below the critical temperature (to the left from λ -line). We would like to point out that our aim is to study behavior of phase transition in liquid helium above critical temperature, where all quantum effects can be neglected. According to the classical work on critical

dynamics [7] to a given static universality class, different dynamic classes can be assigned. At the present time, there is no general consensus which dynamic model (E or F) is genuine from the point of view of experimentally measurable quantities. Both models E and F are developed from model C [7, 39] by adding new interaction terms. Model F reduces to model E as an appropriate coupling constant (g_2 in our notation) equals zero. In the corresponding static model, one of the ω indices coincides directly with the famous, experimentally measurable index α [39]. The index α was calculated in the framework of the renormalization group approach (RG) using resummation procedure [40] up to the four-loop perturbation precision and was measured in the famous Shuttle experiment [41]. The present-day value $\alpha = -0.0127$ is generally accepted. The negativity of the index α ensures $g_2^* = 0$ for the stable fixed point. That means that the stability of model E can be considered as a particular realization of model F.

In vicinity of a critical point the correlation length diverges. Due to the small value of viscosity any small velocity fluctuation can be considerably enhanced and thus large Reynolds number is to be expected. According to Kolmogorov hypothesis [10, 11] there is an inertial interval in which transfer of energy from large to small scales takes place. In this interval homogeneous isotropic turbulence is realized. We would like to make an important note with respect to a turbulent regime. We assume that there is large scale behavior of the fluid near the outer boundary of the system. One can imagine that some large vortex of the system size is created, i.e., that energy is pumped into a system at $L \rightarrow \infty$ scales. The injected energy is then transported via non-linearities in Navier Stokes equation from the outer to smallest scales. This transport takes place in aforementioned inertial interval where scaling behavior is observed [8]. In what follows we employ the perturbative RG technique in order to gain information about these velocity fluctuations on the critical behavior. It is possible to proceed along different ways. One that is more difficult is based on the analysis of composite operators [39]. As was mentioned in [42] composite operators corresponding to the velocity field are very complicated objects already in model E. Also IR irrelevance for model F has been demonstrated only for small values of ϵ . Multiloop calculations of composite operators with subsequent resummation is even more complicated task than the standard calculation of beta functions and critical exponents. After initial efforts in the 1970-80s in the theory of critical phenomena we are not aware of any other work whose aim is similar. It is very probable that this remains so also in the near future. We address this question from a different point of view. Instead of using only one expansion parameter (deviation from upper critical dimension) we introduce an additional one related to the scaling of a fluctuating velocity field. Our approach is motivated by other works [43–45] in which this technique has led to new and interesting results. After successful renormalization we set

expansion parameters to their physical values, which is a common procedure in perturbative RG approach [39, 46]. For a complete solution of our problem it is necessary to perform multiloop calculation and subsequent resummation of diagrams using RG equations. Though feasible and simpler than computation of composite operators, a multiloop calculation is well beyond a scope of this work. At this place it is advantageous to make an important remark. There exist two RG fixed points in the dynamic model E, which are candidates to the possible IR stable regimes (see, e.g., [39]). On one hand, two-loop calculations [47] do not lead to the decision how to choose the true fixed point because of the lack of accuracy in the ω calculation. On the other hand, the value of the ω index depends on the chosen dynamic model. For the same fixed point the ω value obtained in the framework of model F can differ from the analogous ω in model E. Moreover, an inclusion of hydrodynamic fluctuations can enhance this difference.

Model E with activated hydrodynamic modes was proposed and investigated by the RG method in [48]. Particularly, hydrodynamic modes were shown to give significant contributions to the ω index, which is crucial for a general stability analysis. Therefore, an investigation of the most general model F extended by velocity fluctuations is highly actual and desired. By the word general we would like to stress that model F contains all possible relevant terms, in which all IR irrelevant terms have been dropped.

This work is organized as follows. In Section II, we overview model F in the framework of the microscopic description. In Section III, the dynamic equations for the most general case are derived. The stochastic model given by these equations, stochastic Navier-Stokes equation, and suitable asymptotic and retardation conditions are reformulated as an effective field-theoretic model with Martin-Siggia-Rose action [49] and subsequently analyzed. The ultraviolet (UV) renormalization of the model and the elaborated algorithm for the calculation of the renormalization constants are described in Section IV. The fixed points of the renormalization group approach (RG) are calculated and classified together with their stability regions and possible scaling regimes in Section V. The conclusions and results of the one-loop calculations of Feynman graphs are presented in Sections VI and VII, respectively.

II. MICROSCOPIC BACKGROUND OF THE STANDARD F MODEL

The microscopic background of critical dynamics for superfluid Bose-condensed systems was considered previously in [50]. The large scale effective model and corresponding stochastic equations were described directly in the framework of time dependent Green functions at finite temperature. From a microscopic point of view it was the case of model F that needs activation of hydro-

dynamic modes. In this section, we present the basic arguments in order to account for them. Besides, it is useful to give a physical meaning of the fields and parameters of the model.

Let us start with the action for a quantum-field model [51]

$$S = \psi^+ \left(\partial_\tau - \frac{\Delta}{2m_0} - \mu \right) \psi + \frac{\lambda}{2} (\psi^+ \psi)^2. \quad (1)$$

Here $\psi^+(\mathbf{x}, \tau)$, $\psi(\mathbf{x}, \tau)$ -fields appear as a path integral representation of quantum-field operators for Bose-particles, \mathbf{x} is a d -dimensional coordinate, τ is a complex parameter whose real and imaginary parts are constructed in the time variable and the temperature. The symbol Δ is the coordinate Laplace operator $\partial^2 \equiv \partial \cdot \partial$, m_0 is a particles mass, and μ is a chemical potential. The local "density-density" form of the interaction, whose intensity is given by the coupling constant λ , is a usual but not essential approximation. All necessary integrations in \mathbf{x} and τ are implied. We will not discuss here the integration contour in the τ plane [52] because it is not essential for our analysis.

Let us introduce a cutoff parameter Λ dividing the full momentum space $\mathbf{p} \in \mathbb{R}^d$ into small and large momentum regions, $|\mathbf{p}| < \Lambda$ and $|\mathbf{p}| \geq \Lambda$, respectively. By analogy with the Brownian motion, the large scale phenomena are macroscopic in nature and form a mean field, just as small scale phenomena are responsible for stochasticity. So one divides the initial field variables into the hard and soft components (the momentum representation is assumed)

$$\begin{aligned} \psi(\mathbf{p}, \tau) &= \phi + \xi, \\ \phi &= \psi(\mathbf{p}, \tau) \theta(\Lambda - |\mathbf{p}|), \quad \xi = \psi(\mathbf{p}, \tau) \theta(|\mathbf{p}| - \Lambda) \end{aligned}$$

and analogous relations for ψ^+ field.

The average value of the field is considered to be an order parameter for the given dynamic model of critical phenomena [7, 39]. Let us introduce the notation $\langle \dots \rangle$ to describe averaging with respect to the soft fields with $\exp(-S)$ serving as a distribution function. The averaging with respect to the hard fields ξ , ξ^+ is postponed, the related terms will be responsible for the random force in the stochastic equations. The dynamic equation for the soft mean fields ϕ, ϕ^+ can be written with the help of the Schwinger-Dyson equation

$$\begin{aligned} \left(\partial_\tau + \frac{\Delta}{2m_0} + \mu \right) \langle \phi^+ \rangle &= \lambda \langle \phi^+ \phi \phi^+ \rangle + 2\lambda \langle \phi^+ \rangle \xi \xi^+ \\ &+ \lambda \langle \phi \rangle \xi^+ \xi^+ + \lambda \langle \xi^+ \xi \xi^+ \rangle. \end{aligned} \quad (2)$$

An analogous complex conjugated equation with simultaneous change of the sign at the ∂_τ -term is also satisfied.

We observe that the term $\langle \xi^+ \xi \xi^+ \rangle$ takes a form of an additive random force for the soft fields. Such systems are successfully described by the functional Legendre transformation [53]. Let us introduce α, α^+ fields by $\alpha \equiv \langle \phi \rangle$,

$\alpha^+ \equiv \langle \phi^+ \rangle$ and let $\Gamma(\alpha, \alpha^+)$ be a functional which is obtained by the standard Legendre transformation of the generating functional for connected graphs. In terms of the fields α, α^+ Eq. (2) takes the form

$$\begin{aligned} (\partial_\tau + \Delta/2m_0 + \mu) \alpha^+ &= \lambda \alpha \alpha^+ \alpha^+ + \lambda \xi \xi^+ \xi^+ \\ &+ \lambda (2\alpha^+ \xi \xi^+ + \alpha \xi^+ \xi^+ + 2\mathcal{K}_1 \alpha^+ + \mathcal{K}_2 \alpha + \mathcal{L}), \end{aligned} \quad (3)$$

where the following Feynman diagrams appear

$$\mathcal{K}_1 = \text{loop with cross}, \quad \mathcal{K}_2 = \text{loop with cross and arrow}, \quad \mathcal{L} = \text{triangle with cross and arrow}. \quad (4)$$

In a sense, Eq. (3) is an extended Gross-Pitaevskii equation with a random microscopic force $\lambda \xi \xi^+ \xi^+$ and additional loop terms. Graphs in Eq. (4) are constructed from full vertices denoted as $\Gamma_3 \equiv \delta^3 \Gamma / (\delta \alpha)^3$ with all possible cross symbols arrangements at the varying α fields. The propagator lines are connected graphs of the correlators $\langle \phi \phi \rangle$, $\langle \phi \phi^+ \rangle$, $\langle \phi^+ \phi \rangle$, $\langle \phi^+ \phi^+ \rangle$ with the cross marked ϕ^+ fields. The matrix of propagators is determined by $(-\Gamma_2)^{-1}$, where $(-\Gamma_2)$ is hermitian matrix with the elements

$$\begin{aligned} -\frac{\delta^2 \Gamma}{\delta \alpha^2} &= \lambda \alpha^{+2} + \lambda \xi^{+2} + \text{loop terms}, \\ -\frac{\delta^2 \Gamma}{\delta \alpha^+ \delta \alpha} &= -\partial_\tau - \frac{\Delta}{2m_0} - \mu + 2\lambda m + \text{loop terms}, \end{aligned}$$

where $m \equiv \langle \psi^+ \psi \rangle = \alpha^+ \alpha + \xi^+ \xi + \mathcal{K}_1$. The explicit expressions for the loop contributions can be found in [50].

The field m corresponds to a linear combination of internal energy and density of models E and F [3, 7]. Its dynamic equation has obviously the following form:

$$\partial_\tau m = \langle \psi^+ \partial_\tau \psi + \psi \partial_\tau \psi^+ \rangle = \left\langle \psi \frac{\Delta}{2m_0} \psi^+ \right\rangle - \left\langle \psi^+ \frac{\Delta}{2m_0} \psi \right\rangle$$

that can be rewritten in terms of α^+, α variables as follows:

$$\begin{aligned} 2m_0 \partial_\tau m &= \alpha^+ \Delta \alpha - \alpha \Delta \alpha^+ + \text{loop with cross} - \text{loop with cross and arrow} \\ &+ (\xi^+ \Delta \xi - \xi \Delta \xi^+). \end{aligned} \quad (5)$$

The Δ symbol in the loop contributions denotes the Laplace operator. An analogous equation is fulfilled by the local energy density, i.e., by the quantity $\partial_i \psi^+ \partial_i \psi$. Moreover due to presence of two derivatives one can argue that it is in fact less relevant from the RG point than the field m .

Equations (3) and (5) can be rigorously reduced to the usual stochastic equations of model F [50]. To this end, one needs to consider the loop contributions in a perturbative fashion and expand all obtained diagrams in fields, external momenta and frequencies as in the usual theory

of critical phenomena. In general, the probability distribution of random forces is not Gaussian. Nevertheless, in the critical region it can be reduced to the white noise.

In order to comply with the standard notation of model F, we rename the α , α^+ fields as ψ , ψ^+ .

III. THE ACTION AND DYNAMICS OF MODEL F WITH HYDRODYNAMIC MODES ACTIVATED

A standard way for constructing models of critical dynamics is based on the Poisson bracket construction [3, 7] using the correspondence principle. This method can be used to derive equation of motion for macroscopic observables that have their microscopic counterparts. When this is not the case (e.g. entropy) one must proceed in a different fashion and employ symmetry operations and related group generators to derive the Poisson brackets. In this work we rely on the latter approach, whose details are discussed in [39].

In the terminology proposed in [7] model F of critical dynamics is described by the order parameter of conjugated fields $\psi(\mathbf{x}, t)$, $\psi^+(\mathbf{x}, t)$ that are averages of the Bose-particle field operators, an external magnetic field $h_0(\mathbf{x}, t)$, and a field $m(\mathbf{x}, t)$ connected with temperature fluctuations in the system. The dynamics of all these fields is given by the Langevin equations

$$\begin{aligned}\partial_t \psi &= f_\psi + \lambda_0(1 + ib_0) \frac{\delta S_F}{\delta \psi^+} + i\lambda_0 g_{03} \psi \frac{\delta S_F}{\delta m}, \\ \partial_t \psi^+ &= f_\psi^+ + \lambda_0(1 - ib_0) \frac{\delta S_F}{\delta \psi} - i\lambda_0 g_{03} \psi^+ \frac{\delta S_F}{\delta m}, \\ \partial_t m &= f_m - \lambda_0 u_0 \partial^2 \left(\frac{\delta S_F}{\delta m} \right) + i\lambda_0 g_{03} \\ &\quad \times \left(\psi^+ \frac{\delta S_F}{\delta \psi^+} - \psi \frac{\delta S_F}{\delta \psi} \right).\end{aligned}\quad (6)$$

The static action S_F is defined as

$$\begin{aligned}S_F &= \int d^d \mathbf{x} \int dt \left(\psi^+ \partial^2 \psi - \frac{1}{2} m^2 + m h_0, \right. \\ &\quad \left. - \frac{1}{4} g_{01} (\psi^+ \psi)^2 + g_{02} \psi^+ \psi m \right).\end{aligned}$$

The random forces f_ψ, f_m are assumed to be Gaussian random variables with zero means and correlators D_ψ, D_m with the white-noise correlations in time. Their time-momentum $(t, p \equiv |\mathbf{p}|)$ representation then reads

$$D_\psi(p, t, t') = \lambda_0 \delta(t - t'), \quad D_m(p, t, t') = \lambda_0 u_0 p^2 \delta(t - t'). \quad (7)$$

The constants g_{01}, g_{02} and g_{03} define the intensity of (self)interactions of the order parameter and m field; the parameters λ_0 and u_0 relate to the diffusion coefficient, b_0 is an intermode coupling. All these parameters are marked with the subscript “0” to distinguish them from their renormalized counter-partners below.

At $g_{02} = b_0 = 0$, the set of Eqs. (6) and (7) is transformed into the equations for model E. As stated by

De Dominicis [47] and A. N. Vasil’ev [39] it probably represents the IR stable limit of the initial model F.

At the transition to the superfluid hydrodynamics, i.e. at the limit $\nu_0 \rightarrow 0$, the Reynolds number increases $\text{Re} \equiv LV/\nu_0 \rightarrow \infty$, here ν_0 denotes a coefficient of molecular viscosity, L is an outer length of turbulence, V is a characteristic (mean) velocity. Then, one necessarily meets with a phenomenon of developed turbulence [8, 54]. Unfortunately, in the above models E and F velocity field and viscosity contributions are not taken into account because of their IR irrelevance. The corresponding dynamic effects are not investigated in the vicinity of phase transition point yet. We will return to the IR irrelevance discussion with the canonical dimension analysis below.

The stochastic dynamic model with the hydrodynamic modes activated in the vicinity of the lambda point was proposed in [42]. Random velocity fluctuations around a mean velocity \mathbf{V} are represented by the velocity field $\mathbf{v}(\mathbf{x}, t)$ which is now taken into consideration and assumed to be transversal, $\text{div} \mathbf{v} = 0$. In this paper we restrict our attention to the case of incompressible fluid mainly because under usual circumstances the fluid velocity is much smaller than the sound velocity. In such case [9] the fluid is virtually incompressible. Though it is feasible to include also a longitudinal velocity part [55, 56] into a model, such problem is much more demanding from the computational point of view. Moreover it also brings about other physical effects as sound in turbulent media, shock waves etc. Already for critical dynamics near the liquid-gas transition the elimination of sound modes is not completely trivial [57]. These and related issues are left for future research.

The equations were derived in accordance with the equilibrium static limit [39] and Galilean invariance and can be written in a compact notation

$$\nabla_t \varphi_a = \eta_a + (\alpha_{ab} + \beta_{ab}) \frac{\delta S^{st}}{\delta \varphi_b}, \quad \nabla_t \equiv \partial_t + \mathbf{v} \cdot \partial, \quad (8)$$

$$S^{st} = S_F - \frac{1}{2} \int d^d \mathbf{x} \int dt \mathbf{v}^2,$$

with the set of fields $\varphi_a \in \{\psi, \psi^+, m, \mathbf{v}\}$ and the set of random forces $\eta_a \in \{f_\psi, f_{\psi^+}, f_m, \mathbf{f}_v\}$. The tensor α is a symmetric matrix of Onsager coefficients and tensor β represents an anti-symmetrical matrix of streaming coefficients. According to the definition (see Section 5.9 in [39]), their real-space representation reads :

$$\begin{aligned}\alpha_{ab} &= \begin{pmatrix} 0 & \lambda_0 & 0 & 0 \\ \lambda_0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_0 u_0 \partial^2 & 0 \\ 0 & 0 & 0 & -\nu_0 \partial^2 \end{pmatrix}, \\ \beta_{ab} &= \begin{pmatrix} 0 & i\lambda_0 b_0 & i\lambda_0 g_{03} \psi & \psi \partial \\ -i\lambda_0 b_0 & 0 & -i\lambda_0 g_{03} \psi^+ & \psi^+ \partial \\ -i\lambda_0 g_{03} \psi & i\lambda_0 g_{03} \psi^+ & 0 & m \partial \\ -\psi \partial & -\psi^+ \partial & -m \partial & 0 \end{pmatrix}.\end{aligned}\quad (9)$$

In fact, this is a generalization of the standard Langevin equation (6) due to replacing of the partial derivative ∂_t by the Lagrangian derivative ∇_t . Indeed, the terms $-\partial_i(v_i\psi)$, $\partial_i(v_im)$, $\partial_i(v_iv)$ are essential to account for Galilean invariance, but they have to enter only as inter-mode coupling contributions in $\nabla_t\psi$ and ∇_tm equations. That is exhibited in the last column of the β_{ab} matrix (9). Then, due to the antisymmetry condition on β_{ab} , the terms in equation for ∇_tv require a form corresponding to the last line in β_{ab} [42, 48].

Finally, the dynamic equations (8) have the following form: for ψ field

$$\begin{aligned} \partial_t\psi + \partial_i(v_i\psi) &= f_\psi + \lambda_0(1 + ib_0)[\partial^2\psi \\ &\quad - g_{01}(\psi^+\psi)\psi/3 + g_{02}m\psi] \\ &\quad + i\lambda_0g_{03}\psi[g_{02}\psi^+\psi - m + h_0], \end{aligned} \quad (10)$$

for m field

$$\begin{aligned} \partial_tm + \partial_i(v_im) &= f_m - \lambda_0u_0\partial^2[g_{02}\psi^+\psi - m + h_0] \\ &\quad + i\lambda_0g_{03}[\psi^+\partial^2\psi - \psi\partial^2\psi^+], \end{aligned} \quad (11)$$

for the velocity field v

$$\begin{aligned} \partial_tv + \partial_i(v_iv) &= \mathbf{f}_v + \nu_0\partial^2\mathbf{v} \\ &\quad - c\psi^+\partial[\partial^2\psi - g_{01}(\psi^+\psi)\psi/3 + g_{02}m\psi] \\ &\quad - c\psi\partial[\partial^2\psi^+ - g_{01}(\psi^+\psi)\psi^+/3 + g_{02}m\psi^+] \\ &\quad - cm\partial[g_{02}\psi^+\psi - m + h_0] \end{aligned} \quad (12)$$

and equation for ψ^+ field is given by complex conjugation of Eq. 10. The model is extended here by the parameter c in the last equation. It turns out to be convenient in the IR analysis below. The substitution $c = 1$ corresponds to the original stochastic problem (8).

The noise correlator of the force \mathbf{f}_v can be expressed in the form

$$D_v(p, t, t') = g_{04}\nu_0^3 p^{\epsilon-\delta} \delta(t - t') \quad (13)$$

in the space with dimension $d = 4 - \epsilon$. The additional exponent δ allows for a deviation from the Kolmogorov turbulent regime [8, 58].

In fact, there are two physically possible and interested regimes. The first one is the regime with hydrodynamic fluctuations near thermodynamic equilibrium that corresponds to the values $\epsilon = 1$, $\delta = -1$, $g_{04} = 1/\nu_0^2$. The second one is the Kolmogorov turbulent regime with $\epsilon = 1$, $\delta = 4$. In this case the noise (13) imitates the energy injection to the system from a range of the largest eddies [10, 11, 59], the constant g_{04} can be interpreted as an energy dissipation rate per unit mass (see, e.g. [58]) and can be measured experimentally. The most advanced approach to the study of developed turbulence is an investigation of its universal characteristics in the inertial interval, that is an intermediate interval of wave numbers (see, e.g. [54]).

Let us discuss the IR irrelevance of hydrodynamic modes (see [7, 47]). The stochastic problem described by the set of equations (8) with η -force noise (7), (13)

$$\langle \eta_a \eta_b \rangle = D_{ab}, \quad D_{ab} = \begin{pmatrix} 0 & \lambda_0 & 0 & 0 \\ \lambda_0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda u_0 \partial^2 & 0 \\ 0 & 0 & 0 & D_v \end{pmatrix}$$

can be transformed into the field theoretic model by the means of the Martin-Siggia-Rose (MSR) mechanism [49] with the Dominicis-Janssen action

$$S = \varphi'_a D_{ab} \varphi'_b + \varphi'_a \left(-\nabla_t \varphi_a + (\alpha_{ab} + \beta_{ab}) \frac{\delta S^{st}}{\delta \varphi_b} \right), \quad (14)$$

where each φ_a field gets a complementary field $\varphi'_a \in \{\psi^+, \psi', m', v'\}$ and the proper equation of the system (8) stands in large brackets. Auxiliary φ'_a fields in formula (14) appear in the MSR transformation procedure as in the usual case of stochastic dynamic models. These new fields are interpreted as response field variables [60, 61]. Thus, the constructed action is Gaussian with respect to φ'_a fields, the first term in (14) describes the contributions of all stochastic noises. All necessary integrations over chosen variables (space and time, wave vectors and frequency or combined cases) and sum over vector indices are implicitly assumed.

The standard calculation [39, 46] of canonical dimensions for all fields and parameters of the action S is the most straightforward way to analyze IR irrelevance. The canonical dimensions of the model are presented in Table I. The momentum dimension d^p and the frequency one d^ω can be determined independently. The total dimension $d = d^p + 2d^\omega$ is determined due to the dispersion relation $i\omega \sim p^2$ between frequency ω and the momentum vector \mathbf{p} .

F	$p, 1/x$	$\omega, 1/t$	ψ, ψ^+	$\psi', \psi^{+'}$	m, m'	v	v'
d_F^p	1	0	$\frac{d}{2} - 1$	$\frac{d}{2} + 1$	$\frac{d}{2}$	-1	$d + 1$
d_F^ω	0	1	0	0	0	1	-1
d_F	1	2	$\frac{d}{2} - 1$	$\frac{d}{2} + 1$	$\frac{d}{2}$	1	$d - 1$
F	λ_0, ν_0	u_0, b_0	g_{01}	g_{02}, g_{03}	g_{04}	c	
d_F^p	-2	0	$4 - d$	$\frac{4-d}{2}$	δ	$-(2 + d)$	
d_F^ω	1	0	0	0	0	2	
d_F	0	0	$4 - d$	$\frac{4-d}{2}$	δ	$2 - d$	

Table I. Canonical dimensions of the fields and parameters for model F with hydrodynamic modes activated.

The first term of action (14) at $\varphi'_a = \mathbf{v}'$ corresponds to the $\mathbf{v}' D_v \mathbf{v}'$ contribution and represents the influence of random velocity fluctuations on the critical behavior of the system. It is proportional to the constant g_{04} that stands in the noise D_v (13) and mimics stochasticity of the velocity field. In the case of the thermal equilibrium regime, as $\delta = -1$, the canonical dimension of g_{04} is equal to -1 (see Table I). Then the action term discussed is IR irrelevant. In other words, the random velocity fluctuations do not affect large-scale (infrared) asymptotics of all physically relevant and experimentally measurable quantities. Moreover, for all $d > 2$ the canonical dimension of the parameter c is negative. Therefore, all corresponding terms of the action are irrelevant and again can be omitted. As opposed to the situation in Eq. (12), the IR relevant dynamics of the \mathbf{v} field corresponds in this case to the Navier-Stokes equation

$$\partial_t \mathbf{v} + \partial_i (v_i \mathbf{v}) = \nu_0 \partial^2 \mathbf{v}.$$

As a result, the velocity field \mathbf{v} is not stochastic anymore, its role in action (14) is reduced to the role of an external field. It does not affect the critical behavior of the model in full accordance with [7, 47].

To keep in play the stochastic hydrodynamic fluctuations, we propose a scenario similar to the developed turbulence [43, 62–64] and chemical reaction kinetics [45]. The remedy proposed in [43, 63] is a construction of a double expansion in two small expansion parameters ϵ and δ . In other words it constitutes a generalization of the famous Wilson ϵ -expansion [39, 46, 65]. For the logarithmic theory $\epsilon = \delta = 0$, $d = 4$, all terms of action (14) proportional to c can be omitted as the related canonical dimensions are negative. Hence the IR relevant dynamics of the \mathbf{v} field is the following:

$$\partial_t \mathbf{v} + \partial_i (v_i \mathbf{v}) = \nu_0 \partial^2 \mathbf{v} + f_v, \quad (15)$$

the dynamic equations for the basic fields ψ , ψ^+ , m (10), (11) are not changed in the IR limit. The remaining coupling constants are dimensionless simultaneously at the starting point; the model is logarithmic.

In the following steps one calculates perturbative expansion in ϵ and δ powers.

Note that the viscosity as well as the stochasticity enter the equations in a nontrivial fashion. So the scaling behavior of ν_0 can be analyzed for both thermal equilibrium regime and Kolmogorov regime, respectively. Let us stress that it is not a turbulence of superfluids but a developed turbulence of random medium with the fields ψ , ψ^+ and m emerging in the vicinity of phase transition point as a passive admixture.

IV. RENORMALIZATION AND RENORMALIZATION CONSTANTS

Let us refer to model F with activated hydrodynamic modes as model F_h. Its field-theoretic action reads

$$S = 2\lambda_0 \psi^+ \psi' - \lambda_0 u_0 m' \partial^2 m' + \mathbf{v}' D_v \mathbf{v}'$$

$$\begin{aligned} & + \psi^+ \{ -\partial_t \psi - \partial_i (v_i \psi) + \\ & + \lambda_0 (1 + ib_0) [\partial^2 \psi - g_{01} (\psi^+ \psi) \psi / 3 + g_{02} m \psi] \\ & + i\lambda_0 \psi [g_{07} \psi^+ \psi - g_{03} m + g_{03} h_0] \} \\ & + \psi' \{ -\partial_t \psi^+ - \partial_i (v_i \psi^+) \\ & + \lambda_0 (1 - ib_0) [\partial^2 \psi^+ - g_{01} (\psi^+ \psi) \psi^+ / 3 + g_{02} m \psi^+] \\ & - i\lambda_0 \psi^+ [g_{07} \psi^+ \psi - g_{03} m + g_{03} h_0] \} \\ & + m' \{ -\partial_t m - \partial_i (v_i m) - \lambda_0 u_0 \partial^2 [-m + g_{06} \psi^+ \psi \\ & + h_0] + i\lambda g_{05} [\psi^+ \partial^2 \psi - \psi \partial^2 \psi^+] \} \\ & + \mathbf{v}' \{ -\partial_t \mathbf{v} + \nu_0 \Delta \mathbf{v} - \partial_i (v_i \mathbf{v}) \}, \end{aligned} \quad (16)$$

where the following relations between charges (coupling constants) are fulfilled:

$$g_{05} = g_{03}, \quad g_{06} = g_{02}, \quad g_{07} = g_{02} g_{03}. \quad (17)$$

In the framework of the double (ϵ, δ) expansion the logarithmic theory does not have a static limit which changes the renormalization scheme. We stress that the introduction of the new coupling constants g_{05} , g_{06} , g_{07} restores the multiplicative renormalizability of the model.

The following notes are essential for the renormalization procedure.

The fields ψ , ψ^+ , m can be considered as passive scalars for the \mathbf{v} field because graphs with external \mathbf{v} , \mathbf{v}' lines do not include internal lines of other fields.

Renormalization constants of the terms $\varphi'_a \partial_i (v_i \varphi_a^+)$ and $\varphi'_a \partial_t \varphi_a^+$, for which the generic notation Z is used below, are the same for $\varphi_a = \psi, \psi^+, m$ due to the Galilean invariance. There are no counterterms of the corresponding terms at $\varphi_a = \mathbf{v}$, as in the usual developed turbulence theory. The nonlocal counterterms of $\mathbf{v}' D_v \mathbf{v}'$ type are absent. The renormalization constants of the ψ, ψ^+ fields can be chosen to be real due to the symmetry of the action with respect to a transformation $\{\psi, \psi'\} \rightarrow e^{ir} \{\psi, \psi'\}$, $\{\psi^+, \psi'^+\} \rightarrow e^{-ir} \{\psi^+, \psi'^+\}$, similarly to the static theory [39].

The field theoretic action of the theory (16) obviously has to be real. Introduction of the independent coupling constants g_{03} , g_{06} , g_{07} allows us to make a model multiplicatively renormalizable. All renormalization constants for parameters and fields of the mode are real except $Z_{\psi'}, Z_{\psi'^+}$. It is convenient to express the constant ν_0 as $\nu_0 = u_{01} \lambda_0$ via a new dimensionless coupling constant u_{01} . The relations between bare and renormalized constants then read

$$\begin{aligned} \lambda_0 &= \lambda Z_\lambda, & \nu_0 &= \nu Z_\nu, \\ u_0 &= u Z_u, & b_0 &= b Z_b, \\ g_{01} &= g_1 \mu^\epsilon Z_{g_1}, & g_{02} &= g_2 \mu^{\epsilon/2} Z_{g_2}, \\ g_{03} &= g_3 \mu^{\epsilon/2} Z_{g_3}, & g_{04} &= g_4 \mu^\delta Z_{g_4}. \end{aligned} \quad (18)$$

Renormalization of the fields is achieved through the replacements

$$\psi \rightarrow \psi Z_\psi, \quad \psi_+ \rightarrow \psi_+ Z_{\psi^+},$$

$$\begin{aligned}
\psi' &\rightarrow \psi' Z_{\psi'}, & \psi^{+'} &\rightarrow \psi^{+'} Z_{\psi^{+'}}, \\
m &\rightarrow m Z_m, & m' &\rightarrow m' Z_{m'}, \\
v &\rightarrow v Z_v, & v' &\rightarrow v' Z_{v'}.
\end{aligned} \tag{19}$$

The renormalization constants Z are calculated in the MS scheme [46]. In this scheme they can be represented in the form $Z = 1 + [Z]$ where $[Z]$ denotes the pole part in arbitrary linear combinations of the parameters ϵ and δ . In this notation the computation of the following counterterms:

$$\begin{aligned}
&2\lambda[Z_1]\psi^{+'}\psi', & -\lambda u[Z_2]m'\partial^2 m', & -[Z_3]\psi^{+'}\partial_t \psi, \\
&+\lambda[Z_4]\psi^{+'}\partial^2 \psi, & -\lambda[Z_5]\psi^{+'}(\psi^+ \psi)\psi/3, & \lambda[Z_6]\psi^{+'}m\psi, \\
&-[Z_3]^*\psi'\partial_t \psi^+, & -\lambda[Z_5]^*\psi'(\psi^+ \psi)\psi^+/3, & \lambda[Z_4]^*\psi'\partial^2 \psi^+, \\
&\lambda[Z_6]^*m'\psi^+, & \lambda[Z_7]m'\partial^2 m, & \lambda m'\psi^+[Z_8]\psi, \\
&\mathbf{v}'\nu[Z_9]\Delta v. & &
\end{aligned} \tag{20}$$

is necessary, where we have introduced counterterms $[Z_i]$ (index i in the following will always run from 0, 1, ..., 9). Further by comparing expressions (16) and (20) the relations between $[Z_i]$ and the renormalization constants of the model (18), (19) can be derived in a straightforward fashion

$$\begin{aligned}
Z_\lambda Z_{\psi^{+'}} Z_{\psi'} &= 1 + [Z_1], & Z_\lambda Z_u Z_{m'}^2 &= 1 + [Z_2], \\
Z_{\psi^{+'}} Z_\psi &= 1 + [Z_3], \\
Z_\lambda Z_{\psi^{+'}} Z_\psi (1 + ib Z_b) &= (1 + ib) + [Z_4], \\
Z_\lambda Z_{\psi^{+'}} Z_\psi ((1 + ib Z_b) g_1 Z_{g_1}/3 - i g_7 Z_{g_7}) &= (1 + ib) g_1/3 - i g_7 + [Z_5]/3, \\
Z_\lambda Z_m Z_{\psi^{+'}} Z_\psi ((1 + ib Z_b) g_2 Z_{g_2} - i g_3 Z_{g_3}) &= (1 + ib) g_2 - i g_3 + [Z_6], \\
Z_\lambda Z_u Z_{m'} Z_m u &= u + [Z_7], & Z_{m'} Z_m &= 1, \\
Z_\lambda Z_{\psi^{+'}} Z_\psi Z_{m'} (u Z_u (\mathbf{p} + \mathbf{q})^2 g_6 Z_{g_6} - i (q^2 - p^2) g_5 Z_{g_5}) &= u (\mathbf{p} + \mathbf{q})^2 g_6 - i (q^2 - p^2) g_5 + [Z_8], \\
Z_\nu &= 1 + [Z_9], & Z_\nu &= Z_\lambda Z_{u_1}, & Z_{g_4} Z_\nu^3 &= 1.
\end{aligned} \tag{21}$$

In contrast to the renormalization constants for parameters (18) and fields (19) the counterterms $[Z_i]$ can contain both real and imaginary parts. Moreover, the counterterm $[Z_8]$ depends in a nontrivial way on external momenta \mathbf{p} and \mathbf{q} that are carried by the fields $\psi(\mathbf{p})$ and $\psi^+(\mathbf{q})$, respectively. The notation $p = |\mathbf{p}|$, $q = |\mathbf{q}|$ is assumed here.

The Feynman diagrammatic technique is based on the interaction vertices connected by lines (propagators). The propagators of the model have the following form:

$$\begin{aligned}
\Delta_{mm} &= \frac{2\lambda u k^2}{\omega^2 + \lambda^2 u^2 k^4}, & \Delta_{m'm} &= \frac{1}{i\omega + \lambda u k^2} = \Delta_{mm'}^*, \\
\Delta_{vv}^{ij} &= \frac{g_4 \nu^3 k^{\epsilon-\delta} P_{ij}^k}{\omega^2 + \nu^2 k^4}, & \Delta_{v'v}^{ij} &= \frac{P_{ij}^k}{i\omega + \nu k^2} = \Delta_{vv'}^{ij*}, \\
\Delta_{\psi'\psi^+} &= \frac{1}{i\omega + \lambda(1 - ib)k^2} = \Delta_{\psi\psi^+}^*,
\end{aligned}$$

$$\begin{aligned}
\Delta_{\psi^+\psi'} &= \frac{1}{-i\omega + \lambda(1 - ib)k^2} = \Delta_{\psi^+\psi'}^*, \\
\Delta_{\psi\psi^+} &= \frac{2\lambda}{(\omega - i\lambda k^2(1 - ib))(\omega + i\lambda k^2(1 + ib))} = \Delta_{\psi^+\psi}^*.
\end{aligned}$$

The interaction vertices [39] correspond to the vertex factors $V_{\psi^{+'}\psi v}$, $V_{\psi^{+'}\psi^+\psi\psi}$, $V_{\psi^{+'}m\psi}$, $V_{m'mv}$, $V_{m'\psi^+\psi}$, $V_{v'vv}$ plus their complex conjugates. Their explicit form can be easily obtained from the action (16).

A prescription for the multiplicative renormalization is now determined by expressions (21). To the one-loop approximation we obtain the following relations:

$$\begin{aligned}
[Z_\lambda] &= \text{Re}([Z_4] - [Z_3](1 + ib)), \\
[Z_b] &= [Z_4] - [Z_3](1 + ib) - (1 + ib)[Z_\lambda]/(ib), \\
[Z_{\psi'}] &= ([Z_1] - [Z_\lambda])/2 - i\text{Im}([Z_3]), \\
[Z_\psi] &= [Z_3] - [Z_{\psi'}]^*, \\
[Z_u] &= [Z_7] - [Z_\lambda], \\
[Z_{m'}] &= -[Z_m] = ([Z_2] - [Z_\lambda] - [Z_u])/2, \\
g_1[Z_{g_1}] &= 3\text{Re}\left([Z_5] - ([Z_\lambda] + 2[Z_\psi] + [Z_3]) \right. \\
&\quad \left. \times ((1 + ib)g_1/3 - i g_7) - ib[Z_b]g_1/3\right), \\
g_7[Z_{g_7}] &= -\text{Im}\left([Z_5] - ([Z_\lambda] + 2[Z_\psi] + [Z_3]) \right. \\
&\quad \left. \times \{(1 + ib)g_1/3 - i g_7) - ib[Z_b]g_1/3 \right. \\
&\quad \left. - ib[Z_{g_1}]g_1/3\}\right), \\
g_2[Z_{g_2}] &= \text{Re}([Z_6] - ((1 + ib)g_2 - i g_3)([Z_\lambda] + [Z_m] + [Z_3])), \\
[Z_{u_1}] &= [Z_\nu] - [Z_\lambda], \\
g_3[Z_{g_3}] &= -\text{Im}([Z_6] - ((1 + ib)g_2 - i g_3)([Z_\lambda] + [Z_m] + [Z_3]) - ib g_2[Z_{g_2}] - ib g_2[Z_b])), \\
u g_6[Z_{g_6}] &= g_6 u (2[Z_\psi] - [Z_\lambda] - [Z_{m'}] - [Z_u]) - [Z_8]/4/q^2|_{p=q}, \\
[Z_{g_4}] &= -3[Z_\nu], \\
g_5[Z_{g_5}] &= -g_5([Z_\lambda] + [Z_{m'}] + 2[Z_\psi]) - i\partial_p[Z_8]|_{q=-p/(2p)}, \\
[Z_\nu] &= [Z_9].
\end{aligned} \tag{22}$$

The contributions of all graphs to the Z_i constants are collected in Appendix. Graphs 1 and 2 contribute to Z_1 . Graphs 3 and 4 contribute to Z_2 . Expression for Z_3 is a sum of contributions of 5-7 marked with symbol ω . The contributions of the graphs № 5-7 to Z_4 are marked by p^2 ; Z_5 is a sum of contributions of № 8-29; Z_6 corresponds to № 30-34, Z_7 - № 35-40; Z_8 - № 41-47; Z_9 - № 48, 49.

V. FIXED POINTS

The anomalous dimensions γ of the renormalization group equation (γ functions henceforth) are defined as

follows [39, 46]:

$$\gamma_\alpha = - \sum_i e_{g_i} \partial_{g_i} [Z_\alpha], \quad i = 1, 2, \dots, 7, \quad (23)$$

where $\alpha \in \{\psi, \psi', m, m', v', v', g_i, u, u_1, \lambda, \nu, b\}$.

It is appropriate to rescale the coupling constants

$$\begin{aligned} g_i / (8\pi^2) &\rightarrow g_i \quad \text{if } i = 1, 4, 7; \\ g_i / \sqrt{8\pi^2} &\rightarrow g_i \quad \text{if } i = 2, 3, 5, 6. \end{aligned} \quad (24)$$

Unfortunately, the one-loop approximation for renormalization constants as well as for RG functions yields expressions that are too large to be published in this paper. Even the truncated system for model E_h (model F_h at $b = g_2 = g_6 = g_7 = 0$) published in [48] yields very cumbersome renormalization constants. The corresponding anomalous dimensions are then

$$\begin{aligned} \gamma_\lambda &= \frac{3g_4u_1^2}{8(1+u_1)} + \frac{g_3^2}{(1+u)^3} + \frac{g_3g_5u(2+u)}{(1+u)^3}, \\ \gamma_u &= -\frac{g_3^2}{(1+u)^3} - \frac{g_3g_5(u^3+u^2-3u-1)}{2u(1+u)^3} \\ &\quad + \frac{3g_4u_1^2(1+u_1-uu_1-u^2)}{8u(1+u_1)(u+u_1)}, \\ \gamma_{g_3} &= -\frac{3g_4u_1^2}{8(1+u_1)} - \frac{g_3^2}{(1+u)^3} + \frac{g_5^2}{4u} \\ &\quad - \frac{g_3g_5(1+3u+11u^2+5u^3)}{4u(1+u)^3}, \\ \gamma_{g_5} &= -\frac{3g_4u_1^2(1+2u+2u_1)}{8(1+u_1)(u+u_1)} + \frac{g_3^2(2+9u+3u^2)}{2(1+u)^3} \\ &\quad - \frac{g_3g_5(5u+23u^2+9u^3-1)}{4u(1+u)^3} - \frac{g_5^2}{4u}, \\ \gamma_{g_1} &= -\frac{3g_4u_1^2}{4(1+u_1)} - \frac{5g_1}{3} - \frac{6g_3^2g_5(g_3-g_5)}{ug_1(1+u)} \\ &\quad + \frac{2g_3(1+3u+u^2)(g_3-g_5)}{(1+u)^3}, \\ \gamma_{u_1} &= -\frac{g_3^2}{(1+u)^3} - \frac{g_3g_5u(2+u)}{(1+u)^3} + \frac{g_4(1+u_1-3u_1^2)}{8(1+u_1)}, \\ \gamma_m &= \frac{g_3g_5}{4u} - \frac{g_5^2}{4u}, \quad \gamma_{m'} = -\frac{g_3g_5}{4u} + \frac{g_5^2}{4u}, \\ \gamma_\psi &= \gamma_{\psi^+} = \frac{3g_4u_1^2}{16(1+u_1)} - \frac{g_3(g_3-g_5)(2+4u+u^2)}{2(1+u)^3}, \\ \gamma_{\psi'} &= \gamma_{\psi^{++}} = -\frac{3g_4u_1^2}{16(1+u_1)} + \frac{g_3(g_3-g_5)u(2+u)}{2(1+u)^3}, \\ \gamma_\nu &= \frac{g_4}{8}, \quad \gamma_{g_4} = \frac{3g_4}{8}. \end{aligned}$$

A misprint in [48] is corrected here.

As for RG -functions there are ten β -functions

$$\beta_\kappa = \kappa(-e_\kappa - \gamma_\kappa), \quad \kappa \in \{g_i, u, u_1\},$$

where the canonical dimensions $e_{g_1} = e_{g_7} = \epsilon$, $e_{g_2} = e_{g_3} = e_{g_5} = e_{g_6} = \epsilon/2$, $e_{g_4} = \delta$, $e_u = e_{u_1} = 0$ correspond

to the dimensions in Table I and formula (17). The system of equations

$$\beta_\kappa = 0, \quad \kappa \in \{g_i, u, u_1\} \quad (25)$$

has about 2^{10} different solutions; in principle each of them corresponds to a fixed point. The stability of a point is determined by the set of eigenvalues ω for the first derivative matrix $\Omega = \{\Omega_{ik} = \partial\beta_i/\partial g_k\}$, here β is the full set of β_i functions and g is the full set of charges, $i, k \in \{g_i, u, u_1\}$. The IR asymptotic behavior is governed by the IR stable fixed points with a positive-definite Ω matrix.

It is important that the stability analysis yields different results as one takes into consideration a different number of the perturbation order. For example, in the standard model F it was shown [39, 47] that the one-loop results did not lead to the correct IR fixed point.

The majority of the fixed points can be found only by the numerical calculations. Some fraction of them can be immediately discarded, because they fall out of the region with admissible values for physical parameters. The calculation of a full solution for the system (25) has no sense because of the stability problem discussed below. This is why we have attempted to investigate the system specifically in the different regimes, rather than solving it. Furthermore, we reduce the model with respect to Table II in order to discuss the relationship between different models of critical dynamics.

Standard model F	Model E _h	Standard model E
$g_4 = 0$	$g_2 = g_6 = g_7 = 0$	$g_2 = g_4 = g_6 = g_7 = 0$
$u_1 = 0$	$b = 0$	$u_1 = b = 0, g_3 = g_5$

Table II. Relationship between different models of critical dynamic.

Regarding searching for solutions of RG equations we would like to make the following comment. From the numerical point of view it is easier to look for IR stable points, because solutions of flow equations (given by Gell-Mann-Löw equations) directly flow into the stable fixed points. However, it is practically impossible to determine unstable regimes in this way. Obviously, analytical solutions are of decisive importance.

A. Turbulent scaling regime, $\epsilon = 1$, $\delta = 4$

In this regime the numerical analysis reveals an IR stable fixed point

$$\begin{aligned} g_{4*} &= 10.(6), \quad u_* = 1, \quad u_{1*} = 0.7675919, \\ b_* &= g_{1*} = g_{2*} = g_{3*} = g_{5*} = g_{6*} = g_{7*} = 0. \end{aligned} \quad (26)$$

The one loop approximation at this fixed point gives the following anomalous dimensions γ_i and eigenvalues of the

Ω matrix:

$$\gamma_\nu = \gamma_\lambda = 1.(3), \quad \gamma_\psi = -\gamma_{\psi'} = 1.(3), \quad \gamma_m = \gamma_{m'} = 0, \\ \omega = \{2.087, 1.666, 0.833, 4, 2.921\}. \quad (27)$$

As was mentioned above, the many-loop calculations could change the stability of the fixed points. The fixed points of model E_h turn out to be unstable in the context of model F_h , but this instability could appear at the one-loop approximation only. Then let us include the fixed points of E_h model into consideration and overview them.

The fixed points of model E_h were published in [48]; the stable fixed points are listed in Tab. III, the unstable ones in Tab. IV.

FP	FP1	FP2	FP3	FP4
g_1	0	0	$\frac{3}{5}\varepsilon$	$\frac{3}{5}\varepsilon$
g_3	0	0	$\varepsilon^{1/2}$	$\varepsilon^{1/2}$
g_5	0	0	$\varepsilon^{1/2}$	$\varepsilon^{1/2}$
g_4	0	$\frac{8\delta}{3}$	0	$\frac{8\delta}{3}$
u	0	1	1	1
u_1	0	$\frac{1+\sqrt{13}}{6}$	0	0

Table III. Stable fixed points for model E_h .

The detailed analysis of these points can be found in [48] as well as the discussion about the IR stabilizing influence of the $g_1\psi^{+\prime}\psi^{+2}$ term and destabilizing contributions of the velocity fluctuations.

The charges u and u_1 are not expansion parameters. In fact they denote so-called non-perturbative charges. As shown in previous works [47, 66] from the RG perspective such parameters can also acquire infinite values in the fixed point and there are no inconsistencies within perturbation theory. From a physical point of view it is actually necessary to consider also a limiting case as $u \rightarrow \infty$ or $u_1 \rightarrow \infty$ because such regimes can be possible candidates for a stable point of model E. Therefore, it

FP	FP5	FP6	FP7	FP8	FP9
g_1	0	$\frac{3\varepsilon-2\delta}{5}$	$\frac{3\varepsilon-2\delta}{5}$	0	0
g_3	0	0	0	$\varepsilon^{1/2}$	$\varepsilon^{1/2}$
g_5	$\frac{\sqrt{2(-19+\sqrt{13})\delta+18\varepsilon}}{3}$	$\frac{\sqrt{2(-19+\sqrt{13})\delta+18\varepsilon}}{3}$	0	$\varepsilon^{1/2}$	$\varepsilon^{1/2}$
g_4	$\frac{8\delta}{3}$	$\frac{8\delta}{3}$	$\frac{8\delta}{3}$	0	$\frac{8\delta}{3}$
u	1	1	1	1	1
u_1	$\frac{1+\sqrt{13}}{6}$	$\frac{1+\sqrt{13}}{6}$	$\frac{1+\sqrt{13}}{6}$	0	0

Table IV. Unstable fixed points for E_h model.

FP	FP1 ^I	FP2 ^I	FP3 ^I	FP4 ^I	FP5 ^I
g_1	0	$\frac{3\varepsilon}{5}$	$\frac{3\varepsilon}{5}$	$\frac{1}{5}(3\varepsilon-2\delta)$	$\frac{1}{5}(3\varepsilon-2\delta)$
g_3^2/u	0	$\frac{2\varepsilon}{3}$	$\frac{2\varepsilon}{3}$	0	0
g_5^2/u	0	$\frac{2\varepsilon}{3}$	$\frac{2\varepsilon}{3}$	0	$2\varepsilon-2\delta$
g_4	0	0	$\frac{8\delta}{3}$	$\frac{8\delta}{3}$	$\frac{8\delta}{3}$
$1/u$	0	0	0	0	0
u_1	0	0	0	$\frac{1}{6}(1+\sqrt{13})$	$\frac{1}{6}(1+\sqrt{13})$

Table V. Fixed points for model E_h , $u \rightarrow \infty$

FP	FP1 ^{II}	FP2 ^{II}	FP3 ^{II}	FP4 ^{II}	FP5 ^{II}
g_1	0	0	$\frac{3\varepsilon}{5}$	$\frac{3}{5}(\varepsilon-2\delta)$	$\frac{3}{5}(\varepsilon-2\delta)$
g_3	0	0	0	0	0
g_5	0	$\sqrt{2\varepsilon}$	$\sqrt{2\varepsilon}$	0	$\sqrt{2(\varepsilon-4\delta)}$
g_4/u	0	0	0	$\frac{8\delta}{3}$	$\frac{8\delta}{3}$
u	0	1	1	1	1
$1/u_1$	0	0	0	0	0

Table VI. Fixed points for model E_h , $u_1 \rightarrow \infty$

seems reasonable to consider specific limits as their values tend to infinity. It yields additional fixed points. In the case of $u \rightarrow \infty$, the fixed points obtained are collected in Table V. The case $u_1 \rightarrow \infty$ (case II) are presented in Table VI. The case when both charges u and u_1 tend to infinity simultaneously can be found in Table VII.

There exists a more nontrivial fixed point of model E_h when all charges obtain nonzero values. The related exact expression was not calculated in [48] because the fixed point and its stability region depend on the ε/δ ratio and this makes the γ structure very inconvenient for analytical treatment. Our direct numerical calculation in the turbulent regime yields now the unstable point with

FP	FP1 ^{III}	FP2 ^{III}	FP3 ^{III}	FP4 ^{III}	FP5 ^{III}
g_1	0	0	$\frac{3\varepsilon}{5}$	$\frac{3}{5}(\varepsilon-2\delta)$	$\frac{3}{5}(\varepsilon-2\delta)$
f_3	0	$\frac{2\varepsilon}{3}$	$\frac{2\varepsilon}{3}$	0	0
f_5	0	$\frac{2\varepsilon}{3}$	$\frac{2\varepsilon}{3}$	0	$2(\varepsilon-3\delta)$
f_4	0	0	0	$\frac{8\delta}{3}$	$\frac{8\delta}{3}$
w	0	0	0	0	0
w_1	0	0	0	0	0

Table VII. Fixed points for model E_h , $u, u_1 \rightarrow \infty$

the following location:

$$\begin{aligned} u_* &= 0.756, & u_{1*} &= 0.833, & g_{1*} &= 4.808, \\ g_{3*} &= 1.449, & g_{5*} &= -1.021, & g_{4*} &= 32/3, \end{aligned}$$

and ω indices

$$\omega \in \{-12, 984, 3.646, -1.889, -2.328 \pm .257i, 4\}.$$

Need to say that the corresponding nontrivial fixed point with a physically consistent value ($g_4 > 0$) is absent in the equilibrium regime for model E_h .

B. Thermal equilibrium regime, $\epsilon = 1$, $\delta = -1$

In this regime the numerical analysis of model F_h has not exhibited the existence of the IR stable fixed points of the system (25). Apparently, this is not a physical result as the one-loop approximation is not sufficient in this case. For example, the system (25) reduced to the standard model F leads to the stable fixed point $u_* = 1.366$, $b_* = 0.655\epsilon$, $g_{1*} = 1.199\epsilon$, $g_{2*} = 0.447\epsilon$, $g_{3*} = 1.280\epsilon$. Similarly, the four-loop calculations in the static model C prove the stable scaling regime of the standard model F corresponds to model E with $b_* = 0$, $g_{2*} = 0$ [39]. Then we can state that the multi-loop calculations are necessary to make relevant conclusions.

Dynamic Eqs. (10),(11) and (15) demonstrate that the basic fields ψ , ψ^+ , m play a role of passive scalars for the hydrodynamic modes. That causes the exact perturbative statements $g_{4*} = 0$, $\gamma_{g_{4*}} = 0$ at the IR stable fixed point for $\delta < 0$. Another exact expression $\gamma_{g_4} = -3\gamma_\nu$ yields the relation $\gamma_{\nu*} = 0$ in this regime. The next explicit formula $\gamma_\nu = \gamma_\lambda + \gamma_{u_1}$ leads to the relation $\gamma_{\lambda*} = -\gamma_{u_{1*}}$ for the corresponding fixed point. Using the fixed point equation $\beta_{u_1} = u_{1*}\gamma_{u_1} = 0$ one can observe two possibilities. The former one is $\gamma_{\lambda*} = -\gamma_{u_{1*}} = 0$ and the dynamic index z is rigorously equal to 2 and in the latter $u_{1*} = 0$. As renormalization constants depend on the combination $g_4 u_1^2$, in this case the elements of the Ω matrix related to $\partial\beta_k/\partial g_4$ are equal to 0.

Thus, in this case the parameter u_1 does not affect the fixed points and its stability. However, in general the hydrodynamic modes have influence on the stability analysis as they lead to the new multiplicatively renormalized charges g_5, g_6, g_7 . These new charges produce new columns and rows in the Ω matrix and then they are essential in the analysis of the fixed points stability. Let us remind that this analysis is the main problem of model F at this point.

Our most interesting achievement in the equilibrium regime is a new way to analyze the stability for the standard model E. Indeed, some of the fixed points of model E_h presented in Tables III-VII correspond to the standard model E [39]. They must obey $g_4 = u_1 = 0$ and $g_3 = g_5$, in accordance with Table II. Besides, they are stable in the thermal equilibrium regime, i.e. in the region $\epsilon > 0$, $\delta < 0$. It was the points FP3 and FP2^I

that obeyed these constraints. This coincides with a well known two-loop result [39], though it is unknown which of these two points is stable for the standard model E. In the framework of model F_h these two points have the following ω indices:

$$FP3 : \quad \omega \in \{-0.1\epsilon, 0, 0.055\epsilon, 0.25\epsilon, 0.75\epsilon, \epsilon, 1.5\epsilon, 1.92\epsilon, -\delta\},$$

$$FP2^I : \quad \omega \in \{-0.333\epsilon, -0.01\epsilon, -0.05\epsilon, 0.666\epsilon, \epsilon, 1.3\epsilon, 2.15\epsilon, -\delta\}.$$

That means that the point FP3 seems to be more IR stable with respect to the hydrodynamics effects.

VI. SUMMARY AND CONCLUSIONS

In the previous chapters we have shown that all values of the critical exponents are drastically changed as a result of the turbulent background. Specifically, charges of the fields ψ , ψ^+ , m that govern the standard critical behavior vanish at the IR stable critical point in the presence of developed turbulence (see (26)). In other words, the developed turbulence destroys critical fluctuations. This fact has a simple physical explanation. It is well known that critical behavior of the system is accompanied by an unbounded growth of a correlation radius. On the other hand, in the background of developed turbulence, the cascade of the eddies takes place [8]. Due to the decay of large eddies into smaller ones, the kinetic energy is transferred from the largest to the smallest scales and dissipates. Thus it is reasonable to expect that precisely these eddies (and related cascade mechanism) confine the growth of the correlation radius which crucially changes the critical behavior. Moreover, for the above turbulent regime we have calculated the scaling exponent of effective viscosity which turned out to be equal to 4/3 (see (27)) and, therefore, coincides with the well-known fully developed turbulence value.

We have investigated the regime of equilibrium fluctuations, carried out the analysis and classification of the corresponding fixed points and made some assumptions related to their stability taking into account the peculiarities of the extended model F_h . Then, the critical dimension of viscosity vanishes in the regime of equilibrium fluctuations to all orders in the perturbation theory.

Nevertheless, we need to be careful in the analysis of the obtained results. The corollary gained from the analysis of the thermal equilibrium regime suggests that one-loop calculations of models E_h and F_h are not sufficient to make a definite conclusion about the stability of fixed points. Indeed, in the one-loop approximation of model F the fixed point mentioned at the beginning of Section V B is found as IR stable in the equilibrium regime. However, the comparison of the four-loop and five-loop static results [39] with the two-loop expressions in model E [47] yields the conclusion $g_2 = 0$. It was the fixed point of model E that is suitable for description of the true phase transition point. By analogy, next orders of perturbation

corrections can change the sign of the ω index and modify the stability analysis in the presence of the turbulent background.

To make a final conclusion, we assert that the calculations in the regime with the dominance of equilibrium fluctuations seem to be incomplete. For proper analysis of the theory, high-order calculations with inclusion of turbulent fluctuations and consequent resummation procedure are needed. However, even the one-loop approximation leads to 48 Feynman graphs. The related calculations were possible only due to a multiple cross-checking of results within authors' group. The calculations of multi-loop contributions, evidently, require algorithmization of the work.

The corresponding multi-loop algorithms were elaborated in the cycle of articles [67]; however, their applicability is limited by the models without non-perturbative charges like u and u_1 in model F_h . More exactly, the calculations are possible as the fixed points including non-perturbation charges are calculated at a lower order of perturbation theory. Subsequently the obtained values can be used in calculations of Feynman graphs to the next perturbation order.

In addition to the above stated results, the list of fixed points having a chance to become stable in the multi-loop approximation can be considered as a starting point for high-order computer calculations. This information can be considered as an important contribution to the final decision which model (E or F) is suitable for a description of a phase transition to the superfluid state and whether the turbulent background gives a contribution to the experimentally observed quantities.

Acknowledgments

The work was supported by VEGA grant №1/0222/13 of the Ministry of Education, Science, Research and Sport of the Slovak Republic, by the Russian Foundation for Basic Research within the project 12-02-00874-a and by St-Petersburg State University research grant 11.38.185.2014.

We would like to thank Dr. Martin Vala and the project Slovak Infrastructure for High Performance Computing (SIVVP) ITMS 26230120002.

VII. APPENDIX

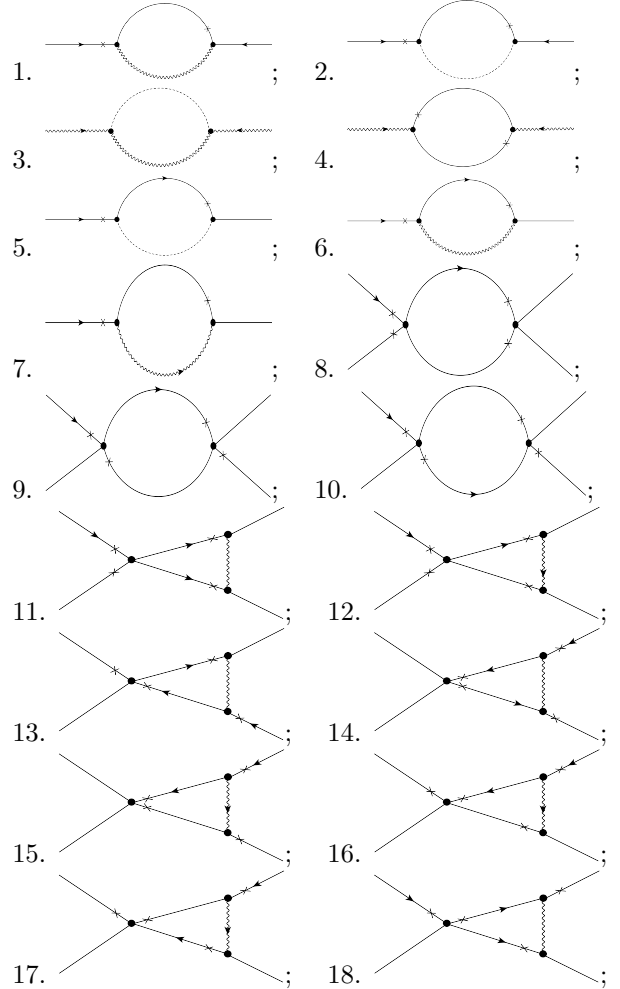
We use the following notation for the vertex factors:

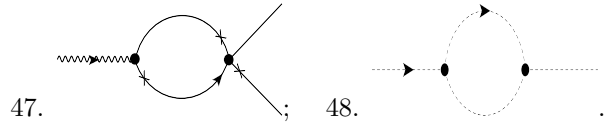
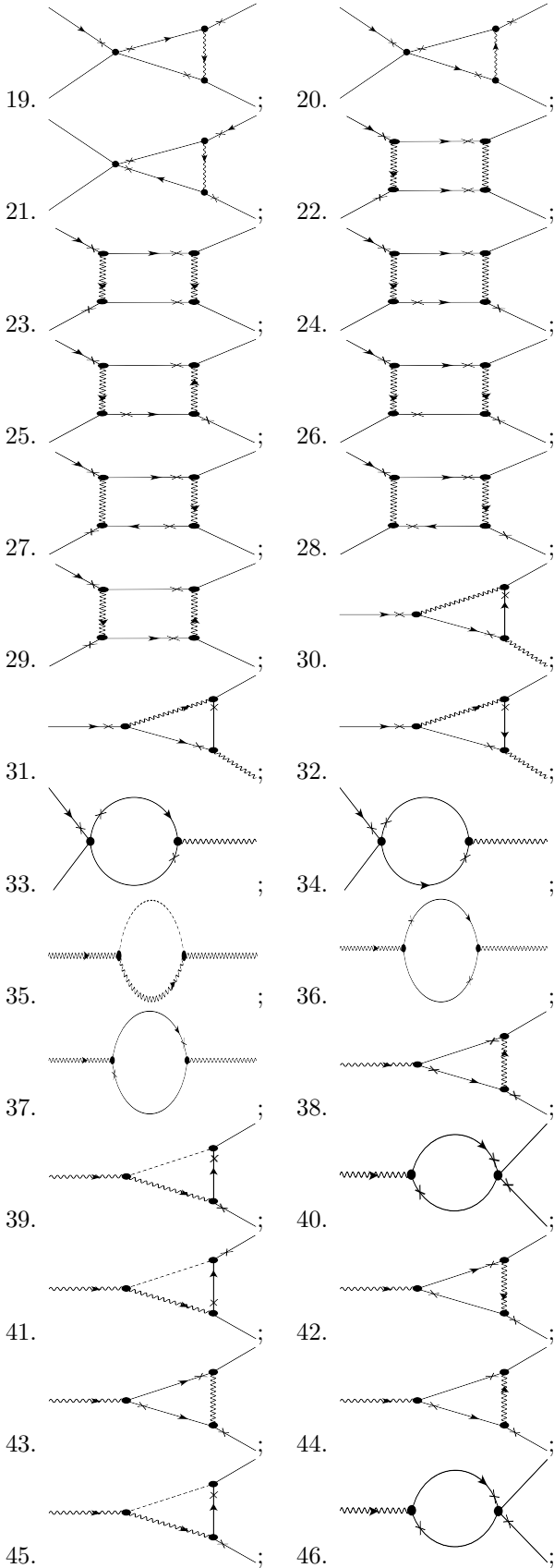
$$\begin{aligned} \begin{array}{c} \psi^+ \\ \times \\ \psi^+ \\ \times \\ \psi^+ \\ \times \\ \psi^+ \end{array} &= -\lambda \left(\frac{g_1}{3} - i \left(\frac{bg_1}{3} - g_2g_3 \right) \right), \\ \begin{array}{c} \psi^+ \\ \times \\ \psi^+ \\ \times \\ \psi^+ \\ \times \\ \psi^+ \end{array} &= -\lambda \left(\frac{g_1}{3} + i \left(\frac{bg_1}{3} - g_2g_3 \right) \right), \end{aligned}$$

$$\begin{aligned} \begin{array}{c} \psi^+ \\ \times \\ \psi^+ \\ \times \\ \psi^+ \\ \times \\ \psi^+ \end{array} &= 1, & \begin{array}{c} \psi \\ \times \\ \psi \\ \times \\ \psi \\ \times \\ \psi \end{array} &= 1, \\ \begin{array}{c} \psi^+ \\ \times \\ \psi^+ \\ \times \\ \psi^+ \\ \times \\ \psi^+ \end{array} &= 1, & \begin{array}{c} m \\ \times \\ m' \\ \times \\ m' \\ \times \\ m' \end{array} &= 1, \\ \begin{array}{c} \psi^+ \\ \times \\ \psi^+ \\ \times \\ \psi^+ \\ \times \\ \psi^+ \end{array} &= \lambda(g_2 - i(bg_2 - g_3)), \\ \begin{array}{c} \psi \\ \times \\ \psi \\ \times \\ \psi \\ \times \\ \psi \end{array} &= \lambda(g_2 + i(bg_2 - g_3)), \\ \begin{array}{c} m \\ \times \\ m' \\ \times \\ m' \\ \times \\ m' \end{array} &= i\lambda g_5(p^2 - q^2) - u\lambda g_6(p + q)^2. \end{aligned}$$

In this graph p and q are the arguments ascribed to $\psi(p)$, $\psi^+(q)$, respectively.

The diagrams are numerated as follows:





The results for the diagrams below contain the following abbreviations

$$\begin{aligned}
 G_{1+} &= ig_7 + g_1(1 - ib)/3, & G_{1-} &= ig_7 - g_1(1 + ib)/3, \\
 G_{2+} &= g_2 + i(bg_2 - g_3), & G_{2-} &= g_2 - i(bg_2 - g_3), \\
 G_{5+} &= ig_5 + ug_6, & G_{5-} &= ig_5 - ug_6, \\
 P &= (\mathbf{p} + \mathbf{q})^2, & Q &= (\mathbf{p} + \mathbf{q}) \cdot \mathbf{q}, \\
 b_+ &= 1 + u + ib, & b_- &= 1 + u - ib, \\
 b_1 &= 1 + ib, & b_{1+} &= 1 + u_1 + ib, \\
 b_2 &= 1 - ib, & b_{2+} &= 1 + u_1 - ib, \\
 u_+ &= u_1 + u,
 \end{aligned}$$

and the coupling constants scaling (24). The results for simple poles of each diagram have the form:

$$\begin{aligned}
 \text{№1} &\Rightarrow -G_{2+}G_{2-}(1 + u)/(b_+b_-\epsilon), \\
 \text{№3} &\Rightarrow -3g_4u_1^2/(8uu_+\delta), \\
 \text{№4} &\Rightarrow -g_5^2/(2u\epsilon), \\
 \text{№5} \sim p^2 &\Rightarrow -3g_4u_1^2/(8b_{1+}\delta), \\
 \text{№6} \sim i\omega &\Rightarrow -G_{2+}^2/(b_+^2\epsilon), \\
 \text{№6} \sim p^2 &\Rightarrow -G_{2+}^2b_1u/(b_+^3\epsilon), \\
 \text{№7} \sim i\omega &\Rightarrow -G_{5+}G_{2+}/(b_+^2\epsilon), \\
 \text{№7} \sim p^2 &\Rightarrow -ig_5G_{2+}[1 + 3u + u^2 + ib \\
 &\quad \times (2 + 3u + ib)]/(b_+^3\epsilon) + ug_6G_{2+}b_1^2/(b_+^3\epsilon), \\
 \text{№8} &\Rightarrow G_{1-}^2/(b_1\epsilon), \\
 \text{№9} &\Rightarrow 2G_{1-}^2/\epsilon, \\
 \text{№10} &\Rightarrow -2G_{1+}G_{1-}/\epsilon, \\
 \text{№11} &\Rightarrow G_{2+}^2G_{1-}/(b_+b_1\epsilon), \\
 \text{№12} &\Rightarrow G_{5+}G_{2+}G_{1-}/(b_+b_1\epsilon), \\
 \text{№13} &\Rightarrow 2G_{2+}^2G_{1-}/(b_+^2\epsilon), \\
 \text{№14} &\Rightarrow G_{2+}G_{2-}G_{1-}/(b_+b_-\epsilon), \\
 \text{№15} &\Rightarrow -G_{5-}G_{2+}G_{1-}(b_+ + 2)/(2b_+b_-b_1\epsilon), \\
 \text{№16} &\Rightarrow G_{5+}G_{2+}G_{1-}(b_+ + 2)/(b_+^2\epsilon), \\
 \text{№17} &\Rightarrow -G_{5+}G_{2+}G_{1+}/(b_+\epsilon), \\
 \text{№18} &\Rightarrow 2G_{1-}G_{2+}G_{2-}(1 + u)/(b_+b_-\epsilon), \\
 \text{№19} &\Rightarrow G_{1-}G_{5+}G_{2-}/(b_+\epsilon), \\
 \text{№20} &\Rightarrow -G_{1-}G_{5-}G_{2+}/(b_-\epsilon), \\
 \text{№21} &\Rightarrow -G_{1-}G_{5-}G_{2+}/(2b_+b_1\epsilon), \\
 \text{№22} &\Rightarrow -G_{5-}G_{2+}^3(u + b_+)/(2ub_+^2b_1\epsilon), \\
 \text{№23} &\Rightarrow -G_{5+}G_{5-}G_{2+}^2(2 + u)/(2ub_+b_-b_1\epsilon), \\
 \text{№24} &\Rightarrow G_{5+}G_{2+}^2G_{2-}(2u^2 + 2u + b_+) \\
 &\quad / (2ub_+b_+^2\epsilon), \\
 \text{№25} &\Rightarrow G_{5+}^2G_{2+}G_{2-}/(2b_+^2\epsilon), \\
 \text{№26} &\Rightarrow -G_{5+}G_{5-}G_{2+}^2[(2 + u)(1 + u) \\
 &\quad + iub]/(2ub_+b_-b_+\epsilon), \\
 \text{№27} &\Rightarrow G_{5+}G_{2+}^2G_{2-}/(2ub_+b_-\epsilon), \\
 \text{№28} &\Rightarrow -G_{5-}G_{2+}^3/(2ub_+^2\epsilon), \\
 \text{№29} &\Rightarrow -G_{5+}G_{5-}G_{2+}^2/(2b_+^2b_1\epsilon), \\
 \text{№30} &\Rightarrow -G_{2+}^3/(b_+^2\epsilon), \\
 \text{№31} &\Rightarrow -G_{5+}G_{2+}^2(2 + b_+)/(2b_+^2\epsilon), \\
 \text{№32} &\Rightarrow -G_{5+}G_{2-}G_{2+}/(2b_+\epsilon), \\
 \text{№33} &\Rightarrow -G_{1-}G_{2-}/\epsilon,
 \end{aligned}$$

$$\begin{aligned}
\text{№34} &\Rightarrow -G_{1-}G_{2+}/\epsilon, \\
\text{№38} &\Rightarrow -(3g_4u_1^2)/(8uu_+\delta), \\
\text{№39} &\Rightarrow ig_5G_{2-}b_1/(4u\epsilon) + g_6G_{2-}/(2\epsilon), \\
\text{№40} &\Rightarrow -ig_5G_{2+}b_2/(4u\epsilon) + g_6G_{2+}/(2\epsilon), \\
\text{№41} &\Rightarrow -3G_{5+}g_4Qu_1^2b_{1+}/\{8u_+\delta[(u_1+1)^2 \\
&\quad +b^2]\}, \\
\text{№42} &\Rightarrow -g_5^2G_{2+}(b_2P/2 - uQ/b_-)/\{2\epsilon \\
&\quad \times(1+u-ib)\} - iug_6g_5G_{2+}P/(2b_-\epsilon) \\
&\quad + u^2g_6^2G_{2+}P/(2b_-\epsilon) - iug_6g_5G_{2+}b_2 \\
&\quad \times(P/2 + Q/b_-)/(2b_-\epsilon), \\
\text{№43} &\Rightarrow -ig_5G_{2+}G_{2-}u[(u^2 + 2u - b^2 + 1)
\end{aligned}$$

$$\begin{aligned}
&\times(P - 2Q) - ib(u^2 + 4u + b^2 + 3)P] \\
&/ (2b_+^2b_-^2\epsilon) + ug_6G_{2+}G_{2-}[(1+u)P]/(b_+b_-\epsilon), \\
\text{№44} &\Rightarrow -g_5^2G_{2-}[b_1P/2 - u(P-Q)/b_+]/(2b_+\epsilon) \\
&\quad + iug_6g_5G_{2-}b_1[P/2 + (P-Q)/b_+]/(2b_+\epsilon) \\
&\quad + iug_6g_5G_{2-}P/(2b_+\epsilon) + u^2g_6^2G_{2-}P/(2b_+\epsilon), \\
\text{№45} &\Rightarrow G_{5-}3g_4u_1^2b_{2+}(P-Q)/(8b_{1+}b_{2+}u_+\delta), \\
\text{№46} &\Rightarrow -ig_5G_{1-}Pb_2/(2\epsilon) + ug_6G_{1-}P/\epsilon, \\
\text{№47} &\Rightarrow -ig_5G_{1+}b_1P/(2\epsilon) - ug_6G_{1+}P/\epsilon, \\
\text{№48} &\Rightarrow -(g_4u_1)/(8\delta).
\end{aligned}$$

The other diagram contributions to the renormalization constants (№2, 35, 36, 37) are equal to zero.

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